PIERCING POINTS OF CRUMPLED CUBES

BY
L. D. LOVELAND

McMillan [20] proved that a free 2-sphere in S^3 can be pierced by a tame arc at each of its points. Since each complementary domain of a free 2-sphere is an open 3-cell [22], it seems natural to attempt to prove some theorem analogous to McMillan's with this weaker hypothesis. We show that a crumpled cube C in S^3 has at most one nonpiercing point if Int C is an open 3-cell. A *crumpled cube* C is the union of a 2-sphere and one of its complementary domains in S^3, and a point p of Bd C is a *piercing point* of C if there is a homeomorphism h of C into S^3 such that $h(\text{Bd } C)$ can be pierced by a tame arc at $h(p)$. It follows that a 2-sphere S in S^3 has at most two points where S cannot be pierced by a tame arc if each component of S^3-S is an open 3-cell (Corollary 3). McMillan [21] obtained the same result independently using an entirely different approach. Our proof follows from Lemmas 1–3 and the main result of [8].

Other results follow from the methods used in the proofs of Lemmas 1–3. For example, if a Cantor set W lies in a 2-sphere S in S^3 such that each component of S^3-S is an open 3-cell, then W is tame (Theorem 3). Thus each Cantor set on a free 2-sphere is tame. We also show that a continuum F is cellular if F lies in the boundary S of a cellular 3-cell and F does not separate S (Theorem 4).

We also obtain two characterizations of piercing points of crumpled cubes. One that is useful in this paper is that a point p in the boundary S of a crumpled cube C is a piercing point of C if and only if Property $(*, p, \text{Int } C)$ is satisfied (Theorem 2). If F is a closed subset of S, $(*, F, \text{Int } C)$ is defined to mean that Bing's Side Approximation Theorem [6] can be applied relative to S and Int C in such a way that the intersection with S of the polyhedral approximation to S lies in the union of a finite set of mutually disjoint small disks in $S-F$. A precise definition can be found in [11] or [15]. The other characterization of piercing points (see Corollary 4) follows as a consequence of this one and results from [17].

The references should be consulted for needed definitions.

I. **Piercing points of the closure of an open 3-cell.**

LEMMA 1. *If* $\varepsilon>0$ *and* p *and* q *are points in the boundary* S *of a crumpled cube* C *in* S^3, *then there exist a crumpled cube* M, *an* ε*-homeomorphism* h *taking* S *onto* Bd M, *and a Sierpinski curve* X *such that*

(1) p *and* q *are inaccessible points of* X,

(2) $X \subset S \cap \text{Bd } M$,

Received by the editors September 12, 1967.

(3) Bd *M is locally tame modulo* $\{p, q\}$,

(4) *M lies in an* ε*-neighborhood of C*,

(5) *h is the identity on X, and*

(6) *each component of* $S-X$ *has diameter less than* ε.

Proof. We shall use the technique introduced by Martin [18]. Let D be a disk on S such that $p \in \operatorname{Int} D$, $q \in S-D$, and Bd D is tame [4], and let $J_1, J_2, \ldots$ be a sequence of tame simple closed curves in D such that Bd $J_1 =$ Bd D and if $D_1, D_2, \ldots$ are the disks on D bounded by $J_1, J_2, \ldots$, respectively, then $D_{i+1} \subset D_i$ and $p = \bigcap D_i$. Using repeatedly the results from [6] and [11] together with the techniques of [4], we obtain a collection of tame annuli $A_1, A_2, \ldots$ such that

Bd $A_i =$ Bd $D_i \cup$ Bd D_{i+1},

$A_i \cap D_i$ contains a Sierpinski curve X_i in $D_i - \operatorname{Int} D_{i+1}$,

$A_j \cap \operatorname{Int} A_i = \varnothing$ if $i \neq j$,

Cl $(\bigcup A_i)$ is a disk E with boundary J_1,

Bd $M = (S-D) \cup E$, and each component of $A_i - X_i$ has diameter less than ε/i.

The procedure for obtaining the annuli A_i is given roughly by Martin in [18], so we do not pursue the details.

The same procedure as outlined above relative to q and the disk $S - \operatorname{Int} D$ yields a sequence of Sierpinski curves Y_i such that $\{p, q\} \cup (\bigcup X_i) \cup (\bigcup Y_i)$ is the Sierpinski curve X required in Lemma 1.

LEMMA 2. *If the closure of* $S^3 - C$ *is a* 3*-cell, then, in addition to the conditions in Lemma* 1, *M and X can be selected such that*

(7) $C \subset M$,

(8) Bd $M \cap S = X$, *and*

(9) *the closure of each component of* $M - C$ *is a* 3*-cell.*

Proof. We use Lemma 1 to obtain a Sierpinski curve X such that X contains p and q inaccessibly, X is locally tame modulo $\{p, q\}$, and each component of $S-X$ is small. Since S is tame from $S^3 - C$ we can obtain a 2-sphere Bd M by pushing each component of $S-X$ slightly into $S^3 - C$. Since Bd M is locally tame modulo X, we see that Bd M is locally tame modulo $\{p, q\}$ [5]. If we identify M as the crumpled cube containing C and bounded by Bd M, conditions (7) and (8) follow. If Z is the closure of a component of $M - C$, then Bd Z is a 2-sphere that is locally tame from Int Z modulo a tame simple closed curve in X. It follows that Bd Z is tame from Int Z (see [11, Theorem 2] and [15, Theorem 14]), so Z is a 3-cell.

LEMMA 3. *Suppose K is a crumpled cube in* S^3 *such that* $S^3 - K$ *is an open* 3*-cell. If* $K_1, K_2, \ldots$ *is a sequence of mutually disjoint* 3*-cells in K such that, for each i,* $K_i \cap$ Bd K *is a disk* D_i, *then* $S^3 - \operatorname{Cl}(K - \bigcup_1^\infty K_i)$ *is an open* 3*-cell.*

Proof. For each integer n we let $M_n = \operatorname{Cl}(K - \bigcup_{i=1}^n K_i)$ and $V_n = S^3 - M_n$. The theorem will follow from [7] once we show that each V_n is an open 3-cell or equivalently that each M_n is cellular. In the remainder of the proof we show that

$M = M_1$ is cellular. Since the K_i are disjoint the same procedure can be used inductively to show the cellularity of each M_n.

Let $\alpha > 0$ and let A be a polyhedral arc in $S^3 - K_1$ from a point k in Int $K - K_1$ to a point m in $S^3 - K$. Since K_1 is a 3-cell, there exist disks D and E such that $D \subset$ Int D_1, Bd D is tame, Bd $D =$ Bd E, $E -$ Bd E lies in Int K_1, and E is homeomorphically within $\alpha/2$ of Bd $K_1 -$ Int D_1. Using Bing's Side Approximation Theorem [6] relative to the open set Bd $K - D$ in Bd K, we obtain an annulus F such that Bd $D \subset$ Bd F, Bd $F -$ Bd $D \subset S^3 - K$, $F \cap (A \cup M \cup \text{Int } E \cup \text{Int } D) = \varnothing$, F lies within $\alpha/2$ of M, and F is locally polyhedral modulo Bd D. Without loss in generality we may assume that $G = F \cup E$ is a polyhedral disk near M [10].

Since $S^3 - K$ is an open 3-cell, there is a 2-sphere R within α of Bd K such that R separates the two boundary components of F, R separates k from m, and R lies in $S^3 - K$. We assume that R is polyhedral [1] and that F and R are in general position. Now we assume that A and R are in general position (i.e., $R \cap A$ consists of a finite number of points where A pierces R), and we choose a component T of $R - F$ such that $T \cap A$ consists of an odd number of points. Then T is a disk with holes lying within α of M. We fill these holes with disks near G to obtain a 2-sphere W that lies within α of M. Since $A \cap W = A \cap T$, it follows that W separates k from m. This means that W bounds a 3-cell X such that $M \subset$ Int X and X lies within α of M.

THEOREM 1. *If C is a crumpled cube in S^3 such that* Int C *is an open* 3-*cell, then C has at most one nonpiercing point.*

Proof. Since there is a homeomorphism h of C into S^3 such that the closure of $S^3 - h(C)$ is a 3-cell [12], [13] and p is a piercing point of C if and only if p is a piercing point of $h(C)$, we assume without loss in generality that the closure of $S^3 - C$ is a 3-cell K. Let p and q be two points of Bd C. We shall show that one of these two points must be a piercing point of C. It follows from Lemma 2 that there exists a 2-sphere S', a Sierpinski curve X, and a crumpled cube M such that $M \subset K$, Bd $M = S'$, p and q are inaccessible points of X, $S' \cap$ Bd $C = X$, S' is locally tame modulo $\{p, q\}$, and the closure of each component of $K - M$ is a 3-cell. From Lemma 3 we see that $S^3 - M$ is an open 3-cell. Since K is a 3-cell it follows that Bd M is locally tame from Int M at both p and q (in fact, M is a 3-cell). For more detail, see the proof of Lemma 5 to follow.

It follows from [8, Theorem 1] that Bd M is also locally tame from $S^3 - M$ at one of the points p and q, say p. Then p lies in a tame arc in X, so p is a point at which Bd C can be pierced by a tame arc [11]. This means that p is a piercing point of C.

Property $(*, p, \text{Int } C)$, which is used in the following theorem, was defined roughly in the introduction and can be found in either [11] or [15].

THEOREM 2. *A point p in the boundary of a crumpled cube C is a piercing point of C if and only if* $(*, p, \text{Int } C)$ *is satisfied.*

Proof. If p is a piercing point of C, there exists a homeomorphism h of C into S^3 such that $h(S)$ can be pierced by a tame arc at $h(p)$. According to Gillman [11] this means that $h(p)$ lies in a tame arc A in $h(S)$. Furthermore Gillman [11] proved that $(*, A, h(\text{Int } C))$ is satisfied since A is tame. Lister [14] showed that $(*, h^{-1}(A), \text{Int } C)$ follows since h is a homeomorphism. Of course this implies $(*, p, \text{Int } C)$.

The other half of the proof of Theorem 2 is merely a rearrangement of the same ideas.

The following result is a consequence of Theorem 2 and a result by Martin [19].

COROLLARY 1. *If p is a point in a 2-sphere S in S^3 and U and V are the components of S^3-S, then either $(*, p, U)$ or $(*, p, V)$ is satisfied.*

COROLLARY 2. *If C and L are the crumpled cubes bounded by a 2-sphere S in S^3 and a point $p \in S$ is a piercing point of both C and L, then S can be pierced by a tame arc at p.*

Proof. It follows from Theorem 2 that both $(*, p, \text{Int } C)$ and $(*, p, \text{Int } L)$ are satisfied. This means that S can be pierced by a tame arc at p [11].

COROLLARY 3. *If each complementary domain of a 2-sphere S in S^3 is an open 3-cell, then S contains two points p and q such that S can be pierced by a tame arc at each point of $S-\{p, q\}$.*

COROLLARY 4. *A point p in the boundary of a crumpled cube C is a piercing point of C if and only if p lies in an arc A in* Bd C *such that for each $\varepsilon>0$ there is a positive number δ such that each unknotted simple closed curve that lies in* Int C *and has diameter less than δ can be shrunk to a point in an ε-subset of S^3-A.*

Proof. From Theorem 2 we see that $(*, p, \text{Int } C)$ is satisfied if p is a piercing point of C. Then p lies in an arc A satisfying the conditions of Corollary 4 (see Theorem 1 in [17] and the remark on p. 511 in [15]).

If there exists an arc A as in the statement of Theorem 2, then $(*, A, \text{Int } C)$ follows (see the remark prior to the statement of Theorem 2 in [17]). Thus it follows from Theorem 2 that p is a piercing point of C.

II. **Certain Cantor sets are tame.** Using the methods of §I, we show that Cantor set W is tame if W lies in a 2-sphere S such that each component of S^3-S is an open 3-cell.

LEMMA 4. *If $\varepsilon>0$ and W is a closed 0-dimensional subset of a 2-sphere S in S^3, then there exists a Sierpinski curve X in S such that*

(1) *W lies inaccessibly in X,*

(2) *X is locally tame modulo W, and*

(3) *each component of $S-X$ has diameter less than ε.*

Proof. The proof is much the same as that given for Lemma 1. We let B_1 be a finite collection of mutually disjoint disks $D_{11}, D_{12}, \ldots, D_{1n_1}$ in S such that each

Bd D_{1j} is tame, $W \subset \bigcap \text{Int } D_{1j}$, and diam $D_{1j} < 1$. We inductively define, for each positive integer n, a similar finite collection B_n of disjoint disks each of diameter less than $1/n$. If we denote the union of the disks in B_n by B_n^*, then we insist that B_n^* is in the interior of B_{n-1}^* and that each point of W is a component of $\bigcap_{n=1}^{\infty} B_n^*$.

Now we apply the procedure outlined in the proof of Lemma 1. First we obtain a tame Sierpinski curve X_1 in $S - \bigcup \text{Int } D_{1i}$ such that each component of $S - X_1$ has diameter less than ε. Then X_2 is a finite collection of tame Sierpinski curves each in the closure of a component of $B_1^* - B_2^*$ and having small holes. This process is continued so that, for each n, we obtain a finite collection X_n of tame Sierpinski curves in the closure of $B_n^* - B_{n+1}^*$. If we denote the union of the Sierpinski curves in X_n by X_n^*, then $(\bigcup X_n^*) \cup W$ is the desired Sierpinski curve X.

LEMMA 5. *If W is a closed 0-dimensional subset of the boundary S of a cellular 3-cell C in S^3 and X is a Sierpinski curve in S containing W such that X is locally tame modulo W, then there exists a cellular 3-cell M such that*

(1) Bd *M is locally tame modulo a point of W,*

(2) $X = \text{Bd } M \cap S$, *and*

(3) $M \subset C$.

Proof. The proof of Lemma 2 shows how to construct a crumpled cube M satisfying conditions (1) through (3), and it follows from Lemma 3 that M is cellular. Since $M \subset C$ and Int C is 1-ULC it is not difficult to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that each simple closed curve lying in Int M and having diameter less than δ can be shrunk to a point in an ε-subset of Int C. This implies Property (A, X, Int M), defined in [15], which is equivalent to ($*$, X, Int M) [15, Theorems 8–10]. Hence M is a 3-cell [15, Theorem 14]. It follows from [8] that Bd M is locally tame modulo a point of W.

THEOREM 3. *If W is a closed 0-dimensional subset of a 2-sphere S in S^3 and each component of $S^3 - S$ is an open 3-cell, then W is tame.*

Proof. Let U and V be the components of $S^3 - S$, and let X be a Sierpinski curve satisfying the conditions of Lemma 4 relative to some $\varepsilon > 0$. There is a homeomorphism h of $S \cup V$ into S^3 such that $S^3 - h(V)$ is a cellular 3-cell [12], [13]. Since $h(X)$ is locally tame modulo $h(W)$ it follows from Lemma 5 that $h(X)$ is locally tame modulo a point $h(p) \in h(W)$. Each Sierpinski curve $h(Y)$ in $h(X) - h(p)$ satisfies ($*$, $h(Y)$, $h(V)$) since $h(Y)$ is tame [11], so it follows from a result by Lister [14] that ($*$, Y, V) is satisfied.

Applying the same argument, where f is a homeomorphism of $S \cup U$ into S^3, we obtain a point $q \in W$ such that each Sierpinski curve Y in $X - \{p, q\}$ satisfies both ($*$, Y, V) and ($*$, Y, U). Then Y is tame [15]. This means that X is locally tame modulo two points, so W is locally tame modulo two points. A theorem proven by Bing [3] shows W to be tame.

COROLLARY 5. *If a closed* 0*-dimensional set* W *lies in the interior of a cellular disk* D *in* S^3, *then* W *is tame.*

Proof. For each point $p \in W$ there exists a disk D_p in Int D and a 2-sphere S_p such that $p \in \text{Int } D_p \subset D_p \subset S_p$ and S_p is locally tame modulo D_p [2, Theorem 5]. It follows from the fact that D_p is cellular [21] that each component of $S^3 - S_p$ is an open 3-cell [23]. This means that W is locally tame (Theorem 3), so W is tame [3].

COROLLARY 6. *If* W *is a closed* 0*-dimensional subset of a free* 2*-sphere* S *in* S^3, *then* W *lies in a tame Sierpinski curve on* S.

Proof. The proof of Theorem 3 shows that W lies in a Sierpinski curve X in S such that X is locally tame modulo two points (each component of a free 2-sphere must be an open 3-cell [22]). Since these two points lie in tame arcs in S (see [20, Theorem 5] and [11, Theorem 6]), they each lie in tame arcs in X [11, Lemma 6.1]. Thus X is tame [10].

III. Other related results.

LEMMA 6. *If* $\varepsilon > 0$ *and* F *is a continuum in the boundary* S *of a crumpled cube* C *in* S^3 *such that* F *does not separate* S, *then there exists a null sequence of mutually disjoint* ε*-disks* $\{E_i\}$ *in* $S - F$ *and a crumpled cube* M *such that*

(1) $F \cup (S - \bigcup \text{Int } E_i) \subset \text{Bd } M$,

(2) Bd M *is locally tame modulo* F, *and*

(3) M *lies in an* ε*-neighborhood of* C.

Proof. Since F does not separate S there is a sequence $\{D_i\}$ of disks on S such that $F = \bigcap D_i$, Bd D_i is tame, and $D_{i+1} \subset \text{Int } D_i$. Now we follow the procedure outlined in the proof of Lemma 1.

LEMMA 7. *If* $\varepsilon > 0$ *and* F *is a continuum in the boundary* S *of a cellular* 3*-cell* C *in* S^3 *such that* F *does not separate* S, *then there exists a cellular* 3*-cell* M *satisfying all the conditions of Lemma* 6 *and such that* $M \subset C$.

Proof. We use Lemma 6 to obtain the disks E_i in $S - F$. Then the construction of M is indicated in the proofs of Lemmas 2 and 3 where each E_i is replaced with a tame disk in C. Since $M \subset C$ and C is a 3-cell it is easy to see that M is also a 3-cell (see the proof of Lemma 5).

THEOREM 4. *If* F *is a continuum on the boundary* S *of a cellular* 3*-cell in* S^3 *such that* F *does not separate* S, *then* F *is cellular.*

Proof. It follows from Lemma 7 that there exists a cellular 3-cell M such that Bd M is locally tame modulo F. Then Theorem 5.2 of [23] insures that F is cellular.

REMARK. The hypothesis that S is the boundary of a 3-cell in the previous theorem cannot be removed. Furthermore, Theorem 4 becomes false if we require only that each component of S^3-S be an open 3-cell. For example, one can grow two of the "feelers" described in [9] into opposite complementary domains of a sphere and let F be any arc containing the two wild points of the resulting sphere. It then follows from [16] that F is not cellular. However, F ought to be cellular if each component of S^3-S is an open 3-cell and F contains at most one (of the two possible) points where S cannot be pierced by a tame arc. Theorem 5 is a special case of this conjecture.

THEOREM 5. *If S is a 2-sphere in S^3 that is locally tame modulo a 0-dimensional set, F is a subcontinuum of S that does not separate S, each component of S^3-S is an open 3-cell, and F contains at most one of the two possible points where S cannot be pierced by a tame arc, then F is cellular.*

Proof. Under the conditions of the hypothesis S can have at most two wild points [8] and F can contain at most one of these points. Then there exists a disk D on S and a point $p \in F$ such that $F \subset \text{Int } D$ and S is locally tame at each point of $D-p$. It follows from Corollary 1 and Theorem 14 of [15] that S is locally tame from one component V of S^3-S at p. If $C=S \cup V$, then C is a cellular crumpled cube. Now we are able to use the technique in the proofs of Lemmas 6 and 7 to obtain a cellular 3-cell M such that $M \subset C$, $D \subset \text{Bd } M$, and $\text{Bd } M$ is locally tame modulo D. Then $\text{Bd } M$ is locally tame modulo F [10], and Theorem 5 follows from Theorem 4.

REFERENCES

1. R. H. Bing, *Approximating surfaces with polyhedral ones*, Ann. of Math. **65** (1957), 456–483.

2. ———, *A surface is tame if its complement is* 1-ULC, Trans. Amer. Math. Soc. **101** (1961), 294–305.

3. ———, *Tame Cantor sets in E^3*, Pacific J. Math. **11** (1961), 435–446.

4. ———, *Each disk in E^3 contains a tame arc*, Amer. J. Math. **84** (1962), 583–590.

5. ———, *Pushing a 2-sphere into its complement*, Michigan Math. J. **11** (1964), 33–45.

6. ———, *Improving the side approximation theorem*, Trans. Amer. Math Soc. **116** (1965), 511–525.

7. M. Brown, *The monotone union of open n-cells is an open n-cell*, Proc. Amer. Math. Soc. **12** (1961), 812–814.

8. C. E. Burgess, *Criteria for a 2-sphere in S^3 to be tame modulo two points*, Michigan Math. J. **14** (1967), 321–330.

9. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional spaces*, Ann. of Math. **49** (1948), 979–990.

10. P. H. Doyle and J. G. Hocking, *Some results on tame disks and spheres in E^3*, Proc. Amer. Math. Soc. **11** (1960), 832–836.

11. D. S. Gillman, *Side approximation, missing an arc*, Amer. J. Math. **85** (1963), 459–476.

12. N. Hosay, *The sum of a real cube and a crumpled cube is S^3*, Notices Amer. Math. Soc. **10** (1963), 668.

13. L. L. Lininger, *Some results on crumpled cubes*, Trans. Amer. Math. Soc. **118** (1965), 534–549.

14. F. M. Lister, *Simplifying the intersection of disks in Bing's Side Approximation Theorem*, Pacific J. Math. **22** (1967), 281–295.

15. L. D. Loveland, *Tame subsets of spheres in* E^3, Pacific J. Math. **19** (1966), 489–517.

16. ———, *Wild points of cellular subsets of* 2-*spheres in* S^3, Michigan Math. J. **14** (1967), 427–431.

17. ———, *Sufficient conditions for a closed set to lie on the boundary of a* 3-*cell*, Proc. Amer. Math. Soc. **19** (1968), 649–652.

18. J. Martin, *Tame arcs on disks*, Proc. Amer. Math. Soc. **16** (1965), 131–133.

19. ———, *The sum of two crumpled cubes*, Michigan Math. J. **13** (1966), 147–151.

20. D. R. McMillan, Jr., *Some topological properties of piercing points*, Pacific J. Math. **22** (1967), 313–322.

21. ———, *Piercing a disk along a cellular set*, Proc. Amer. Math. Soc. **19** (1968), 153–157.

22. C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. **66** (1957), 1–26.

23. D. G. Stewart, *Cellular subsets of the* 3-*sphere*, Trans. Amer. Math. Soc. **114** (1965), 10–22.

Utah State University,
Logan, Utah